ON THE SYNTHESIS OF SPATIAL RACK MECHANISMS: MATHEMATICAL MODELLING AND SOFTWARE TEETH GENERATING OF THE MOVING LINKS TEETH

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Abstract: Spatial three-link rack mechanisms are applied to realize rotation transformation of one of the movable links (pinion) into rectilinear transformation of the other link (gear rack). The principle for the synthesis upon mesh region is applied in this study by developing an adequate mathematical model. Here are shown algorithms, analytically describing active tooth surfaces of rotating link, mesh region and the link which realize translation motion. The results, which treat synthesis of spatial convolute, Archimedean and involute rack drives, are illustrated graphically.

Key words: mathematical modeling, synthesis, rack mechanisms, conic linear helicoid

1. INTRODUCTION

The transformation of the rotation motion into rectilinear translations is a fundamental type motion transformation in the techniques. This fact determines the researches interest in different type mechanical systems, which designation is to ensure the upper mentioned motion transformation, as in quantitative and qualitative terms. Spatial rack mechanisms occupy a special place among these mechanisms [1].

These drive system types can be considered as a special case of spatial three-link mechanisms, transforming rotations between fixed crossed axes, when:

- The number of the teeth of one of the movable link is increased „ad infinitum”, without increasing the number of the meshed tooth surfaces between the mated links;
- the axis of rotation of the upper mentioned link is displaces in infinity and its motion is transformed into rectilinear translation;
- number of the teeth and the type of motion of the second movable link remain unchanged;
- rotating link with finite number of teeth will call pinion, and the link with endless number of teeth, realizing rectilinear translation, will call it gear rack.

This study presents a mathematical model for synthesis upon mesh region [1, 2] of a spatial rack gear set, in which the active tooth surfaces of the pinion are parts of conic linear helicoids, tooth surfaces of gear rack are theoretically conjugated with the those of the pinion. The rotation axis of the pinion is non-orthogonal crossed with the direction of the rectilinear translation of the gear rack.

2. SYNTHESIS OF SPATIAL RACK DRIVES, WHICH ROTATING LINK HAS CONIC LINEAR HELICOIDS

When synthesizing spatial rack mechanisms with linear contact it is evident the necessity to control their quality in the whole mesh region. Such approaches to synthesis problems require an adequate mathematical model. The kinematic scheme of such mechanism is shown in Fig.1[1].

The reason is that the specific geometrical and kinematic characteristics of the mesh region depend on its position in space and the geometrical characteristics of the tool surface $\Sigma_j$ generating the flanks $\Sigma_i$ ($i = 1, 2$).

This mathematical model is based on the second Olivier’s principle and the link, performing rotation and having surfaces $\Sigma_j$ is chosen as a generating link. Generated surfaces $\Sigma_j$ belong to link 2, which realizes rectilinear translation.

The synthesis of spatial rack drives based on the second Olivier’s principle, involves solving of two main tasks:

- Synthesis of the tooth surfaces $\Sigma_j$ of the rotating movable link, which are identical to the instrumental surfaces $\Sigma_j$. Solving this mathematical problem provides the technology of manufacturing the rotating link of the rack drive.
- Synthesis of the mesh region and definition of the dimensions and location of the mesh region on the surface of action. This task realizes the generation of active tooth surfaces $\Sigma_j$ of the movable link of rack mechanisms, which perform motion translation.

The synthesis of these types of mechanical transmissions, in accordance with the second Olivier’s principle, leads to solving two main tasks: synthesis of the active tooth surfaces of the base (instrumental) moving link - conic worm and synthesis of surface of action/mesh region of the studied mechanism. The active tooth surfaces of the basic link are conic linear helicoids.
2.1. Generating of active tooth surfaces of the basic movable link

The helicoids and especially the linear helicoids are widely used as active tooth surfaces of spatial gears with crossed axes, in both working meshing conditions and instrumental meshing conditions.

![Image](image.png)

**Fig.1. Kinematic scheme of spatial rack drive with linear contact between toothed surfaces \( \Sigma_1 \) and \( \Sigma_2 \) synthesized using a mesh region (MR): \( i = 1 \) - pinion (gear) is the rotating link; \( i = 2 \) - rack gear is the link of translation (rectilinear); \( \Sigma_1 \) - surface of flank of links \((i = 1, \ 2)\); \( \overrightarrow{\alpha}_j \) - rotation velocity of link \( i = 1 \); \( \overrightarrow{V}_2 \) - velocity of rectilinear translation of link \( i = 2 \); \( j_{12} = 1/ j_{21} = \alpha_1/V_2 = \text{constant} \) - velocity ratio of motion transformation; \( \overrightarrow{V}_{12} \) - sliding velocity; \( \overrightarrow{AS} \) - action surface; \( \overrightarrow{D}_{12} \) - contact line; \( \overrightarrow{L}_1 \) - longitudinal line of \( \Sigma_1 \).

Such choice of the active tooth surfaces for these gears is determined by the technological manufacturing, especially when their synthesis is performed in accordance with the second Oliver’s principle [2, 3]. In most cases, these surfaces are the only alternative, as it is the manufacture of cylindrical worm gear and Spiroid and Helicon gears.

Fig. 2 shows the generation of right-handed conic convolute helicoids \( \Sigma_i^{(j)} \) \((j = 1, \ 2)\) determined by different geometrical characteristics of the helical teeth (threads) of the particular gear mechanism. The process of helical surfaces generation is considered in fixed coordinate system \( S_p^{(j)}(O_p^{(j)}, x_p^{(j)}, y_p^{(j)}, z_p^{(j)}) \) and it is as follows. The generatrix \( \overrightarrow{L}_i^{(j)} \) does not cross the axis \( O_p^{(j)}z_p^{(j)} \) which coincides with the geometric axis of the gear. \( \overrightarrow{L}_i^{(j)} \) and \( O_p^{(j)}z_p^{(j)} \) conclude an angle \( 0,5\pi < \xi^{(j)} < \pi \). The line \( \overrightarrow{L}_i^{(j)} \) belongs to plane \( T_i^{(j)} \), which is tangential to the directed circle cylinder \( C_i^{(j)} \).

The generation of the conic convolute helicoid \( \Sigma_i^{(j)} \) by the line \( \overrightarrow{L}_i^{(j)} \) is realized by two generatrix motions: axial helical motion along the longitudinal axis \( O_p^{(j)}z_p^{(j)} \) with parameter \( p_s^{(j)} = \text{constant} \); crossed helical motion in the plane \( T_i^{(j)} \), perpendicular to the axis \( O_p^{(j)}z_p^{(j)} \) with parameter \( p_t^{(j)} = \text{constant} \). Here we should note that \( \Sigma_i^{(1)} \) is the conic convolute helical surface that is turned to the positive direction of the axis \( O_p^{(j)}z_p^{(j)} \), and \( \Sigma_i^{(2)} \) is the helicoid, turned along the negative direction of the axis \( O_p^{(j)}z_p^{(j)} \).

The vector equation of the conic convolute helical surface \( \Sigma_i^{(j)} \), in accordance with Fig. 2, has the form \([1, 2, 4]\):

\[
\overrightarrow{R}_i^{(j)} = \overrightarrow{R}_0^{(j)} + \overrightarrow{S}^{(j)} + \overrightarrow{t}^{(j)} + \overrightarrow{u}^{(j)},
\]

where

\( \overrightarrow{R}_i^{(j)} \) is a radius-vector of point \( N_i^{(j)} \) that belongs to the conic convolute helicoid \( \Sigma_i^{(j)} \);

\( \overrightarrow{R}_0^{(j)} \) - radius-vector of the directed cylinder \( C_i^{(j)} \);

\( \overrightarrow{S}^{(j)} \), \( u^{(j)} \) - coordinates of the helical surface \( \Sigma_i^{(j)} \);

\( t^{(j)} = p_t^{(j)} \overrightarrow{g}^{(j)} \) - crossed displacement (tangential to the directed cylinder \( C_i^{(j)} \) of the generatrix line \( L_i^{(j)} \). If we write (1) in coordinate system \( S_p^{(j)} \) we obtain:

\[
x_p^{(j)} = r_0^{(j)} \cos \overrightarrow{g}^{(j)} \pm U^{(j)} \sin \overrightarrow{g}^{(j)}, \\
y_p^{(j)} = r_0^{(j)} \sin \overrightarrow{g}^{(j)} \mp U^{(j)} \cos \overrightarrow{g}^{(j)}, \\
z_p^{(j)} = p_s^{(j)} \overrightarrow{g}^{(j)} \pm u^{(j)} \cos \xi^{(j)}, \\
U^{(j)} = (u^{(j)} \sin \xi^{(j)} - p_t^{(j)} \overrightarrow{g})^{(j)}.
\]

For the equation systems (2) the upper signs and \( j = 1 \) refer to the \( \Sigma_i^{(1)} \), and the lower and \( j = 2 \) refer to the \( \Sigma_i^{(2)} \).

Substituting in (2)

\[
R_0^{(j)} = u^{(j)} - \frac{p_t^{(j)} \overrightarrow{g}^{(j)}}{\sin \xi^{(j)}},
\]
and

\[ p^{(j)} = p^{(j)}_1 \pm p^{(j)}_1 \cot \xi^{(j)}, \]  

(3)

it is obtained

\[ x_p^{(j)} = r^{(j)}_0 \cos \vartheta^{(j)} \pm R^{(j)}_0 \sin \xi^{(j)} \sin \vartheta^{(j)}, \]

\[ y_p^{(j)} = r^{(j)}_0 \sin \vartheta^{(j)} \mp R^{(j)}_0 \sin \xi^{(j)} \cos \vartheta^{(j)}, \]

\[ z_p^{(j)} = p^{(j)} \vartheta^{(j)} \pm R^{(j)}_0 \cos \xi^{(j)}; \]

(4)

Equation (4) represents the conic convolute helical surface \( \Sigma^{(j)} \) as a cylindrical one with helical parameter \( p^{(j)} = \text{constant} \) and coordinates \( \vartheta^{(j)} \) and \( R^{(j)}_0 \). The point \( K^{(j)} \) is the accounting origin of co-ordinate \( R^{(j)}_0 \) \( (K^{(j)} \) is a point of intersection of \( L^{(j)} \) and the generatrix of the directed cylinder \( C^{(j)} \)). \( C^{(j)} \) and plane \( T^{(j)} \) are contacting in this generatrix \( L^{(j)} \). The point \( K^{(j)} \) is considered as a point from the directed helical line \( \vec{p}_0^{(j)} = \vec{p}_0^{(j)}(\vartheta^{(j)}) \) on the \( C^{(j)} \).

It is known [5] that the basic characteristic of the surfaces with straight line generatrices is their parameter of distribution. When the surfaces of this type are generated, the straight line generatrix moves from its initial position to infinite close position, as it is turn at some angle and is displaced to some distance. These two values are infinitely small, but their correlation has a limit, which is called parameter of distribution [9]

\[ h = \lim_{\Delta \varphi \to 0} \frac{\Delta \lambda}{\Delta \varphi} \]  

(5)

To determine the parameter of distribution \( h^{(j)} \) of the conic convolute helical surface \( \Sigma^{(j)} \), the vector equation (1) is presented in the form [2]:

\[ \vec{p}_0^{(j)} = \vec{p}_0^{(j)}(\vartheta^{(j)}) + R^{(j)}_0 \tilde{t}^{(j)}, \]

(6)

where \( \vec{p}_0^{(j)} = \vec{p}_0^{(j)}(\vartheta^{(j)}) \{ \chi_1^{(j)}, \chi_2^{(j)}, \chi_3^{(j)} \} \) is an equation of directed helical line on the directed cylinder \( C^{(j)} \); \( \chi_1^{(j)}, \chi_2^{(j)}, \chi_3^{(j)} \) - projections of vector \( \vec{p}_0^{(j)} \) in coordinate system \( S^{(j)}_p \);

\[ \tilde{t}^{(j)} \{ l_1^{(j)}, l_2^{(j)}, l_3^{(j)} \} \] - directed unit vector of the generatrix \( L^{(j)} \);

\( l_1^{(j)}, l_2^{(j)}, l_3^{(j)} \) - projections of \( \tilde{t}^{(j)} \) in coordinate system \( S^{(j)}_p \).

For the case shown in Fig. 2, we have:

\[ \chi_1^{(j)} = r^{(j)}_0 \cos \vartheta^{(j)}, \quad \chi_2^{(j)} = r^{(j)}_0 \sin \vartheta^{(j)}, \]

\[ \chi_3^{(j)} = p^{(j)} \vartheta^{(j)}, \quad l_1^{(j)} = \pm \sin \xi^{(j)} \sin \vartheta^{(j)}, \quad l_2^{(j)} = \mp \sin \xi^{(j)} \cos \vartheta^{(j)}; \]

\[ l_3^{(j)} = \pm \cos \xi^{(j)}, \]

(7)
Then using (5), as it is shown in [9], for the studied conic convolute helical surface it can be written:

\[ h^{(j)} = \frac{[d\overline{p}^{(j)}][\overline{r}^{(j)}]d\overline{r}^{(j)}}{(dt^{(j)})^2}. \]  

(8)

From (8) for \( h^{(j)} \) it is obtained:

\[ h^{(j)} = p^{(j)} + r_0^{(j)}\cot \xi^{(j)}. \]  

(9)

Conic involute helical surfaces generation. It is known [5] that each developable surface is cylindrical one, conic one or locus of the tangential lines to an arbitrary curve and vice versa - each cylindrical, conic or surface representing the locus of tangential lines to a curve is a developable surface.

The specific characteristic for each developable surface is, that the parameter of distribution is equal to zero, i.e. in the specific case the condition is fulfilled:

\[ h^{(j)} = p^{(j)} + r_0^{(j)}\cot \xi^{(j)} = 0, \]  

or

\[ \cot \xi^{(j)} = -\frac{p^{(j)}}{r_0^{(j)}}. \]  

(10)

After observing the condition (10), the equation (4) describes the conic involute helicoid, which is given in the form:

\[ x_j^{(j)} = r_0^{(j)}\cos \theta^{(j)} \pm R_0^{(j)}\cos \lambda_0^{(j)}\sin \theta^{(j)}, \]
\[ y_j^{(j)} = r_0^{(j)}\sin \theta^{(j)} \mp R_0^{(j)}\cos \lambda_0^{(j)}\cos \theta^{(j)}, \]
\[ z_j^{(j)} = p^{(j)}\theta^{(j)} \mp R_0^{(j)}\sin \lambda_0^{(j)}, \]  

(11)

where \( \lambda_0^{(j)} = \xi^{(j)} - \pi/2 \) is spiral angle [10] for the directed helical line on the cylinder \( C^{(j)} \). For this case the cylinder \( C^{(j)} \) is named basic cylinder.

Conic Archimedean helical surfaces generation. The conic Archimedean helicoid is obtained when the generatrix \( L^{(j)} \) (see Fig. 2) crosses the axis \( O_p^{(j)} z_p^{(j)} \), i.e. when \( r_0^{(j)} = 0 \).

Then from (2) it is obtained:

\[ x_j^{(j)} = \pm U^{(j)} \sin \theta^{(j)} , \]
\[ y_j^{(j)} = \mp U^{(j)} \cos \theta^{(j)} , \]
\[ z_j^{(j)} = p^{(j)}\theta^{(j)} \pm u^{(j)}\cos \xi^{(j)} . \]  

(12)

The curvilinear coordinate \( \theta^{(j)} \) in (12) represents the angle on which rotates the normal vector \( \overline{r}_0^{(j)} \) to the axial plane (determined by the \( L^{(j)} \) and \( O_p^{(j)} z_p^{(j)} \)). If it is noted with \( \vartheta^{(j)} = \vartheta^{(j)} + \pi/2 \), the angle, that the plane \( (L^{(j)}, O_p^{(j)} z_p^{(j)} \) concludes with the plane \( (O_p^{(j)} x_p^{(j)} z_p^{(j)} \), then the equation systems (12) turn into:

\[ x_p^{(j)} = \mp U^{(j)} \cos \vartheta^{(j)} , \]
\[ y_p^{(j)} = \mp U^{(j)} \sin \vartheta^{(j)} , \]
\[ z_p^{(j)} = p^{(j)}(\vartheta^{(j)} - \pi/2) \mp u^{(j)} \cos \xi^{(j)} , \]
\[ U^{(j)} = [u^{(j)} \sin \xi^{(j)} - p^{(j)}(\vartheta^{(j)} - \pi/2)]. \]  

(13)

2.2. Singular points on the conic linear helicoids

Conic convolute helicoids. In order to eliminate the “undercutting” from the active tooth surfaces of the gears and the devices, representing parts of the conic linear helicoids, it is necessary to define conditions for appearing of the singular points of II order (undercutting points) on them in the process of their generation. As it has been already shown, the conic linear helicoids \( \Sigma_i^{(j)} \) (in particular conic convolute helical surface) have the form:

\[ \overline{r}_i^{(j)} = \overline{p}_i^{(j)}(u^{(j)}, \vartheta^{(j)}). \]

Then, the normal vector \( \overline{n}_i^{(j)} \) to the \( \Sigma_i^{(j)} \) in arbitrary point \( N_i^{(j)} \) is determined by equality

\[ \overline{n}_i^{(j)} = \frac{\partial \overline{p}_i^{(j)}}{\partial u^{(j)}} \times \frac{\partial \overline{p}_i^{(j)}}{\partial \vartheta^{(j)}}, \]  

(14)

where \( \frac{\partial \overline{p}_i^{(j)}}{\partial u^{(j)}} \) and \( \frac{\partial \overline{p}_i^{(j)}}{\partial \vartheta^{(j)}} \) are vectors at point \( N_i^{(j)} \) from the conic helical surface \( \Sigma_i^{(j)} \), that are tangential to the coordinate lines \( u^{(j)} = constant \) and \( \vartheta^{(j)} = constant \).

It is known [2, 5], that the point \( N_i^{(j)} \) from \( \Sigma_i^{(j)} \) is an undercutting point, if for this point could not be defined the normal vector \( \overline{n}_i^{(j)} \), i.e. if the following condition is fulfilled:

\[ \overline{n}_i^{(j)} = \overline{0}. \]  

(15)

Then from (2) the projections of the normal vector \( \overline{n}_i^{(j)} \) in the coordinate system \( S_p^{(j)} \) are written of the form:

\[ n_{i_{\overline{p}_p}}^{(j)} = H^{(j)} \cos \vartheta^{(j)} - U^{(j)} \cos \xi^{(j)} \sin \vartheta^{(j)} , \]
\[ n_{i_{\overline{r}_p}}^{(j)} = H^{(j)} \sin \vartheta^{(j)} + U^{(j)} \cos \xi^{(j)} \cos \vartheta^{(j)} , \]
\[ n_{i_{\overline{z}_p}}^{(j)} = U^{(j)} \sin \xi^{(j)} , \]  

(16)

From (16) it is evident, that the condition (15) is never satisfied for the conic convolute helicoid, since \( h^{(j)} \neq 0 \).
This means that the conic convolute helicoid consists of only points that are regularly distributed. Conic involute helicoid. From (16) regarding the condition of distribution \( h^{(j)} = 0 \), for the projections of the normal vector \( \mathbf{n}^{(j)}_1 \) in \( S^{(j)}_p \) at an arbitrary point \( N^{(j)}_1 \) from the conic involute surface \( \Sigma^{(j)}_1 \) is obtained:

\[
\begin{align*}
N^{(j)}_1 &= -U^{(j)} \cos \xi^{(j)} \sin \vartheta^{(j)}; \\
N^{(j)}_2 &= U^{(j)} \cos \xi^{(j)} \cos \vartheta^{(j)}, \\
N^{(j)}_3 &= U^{(j)} \sin \xi^{(j)}.
\end{align*}
\]

(17)

It is evident from (17), that the condition (15) is fulfilled for all points \( N^{(j)}_1 \), belonging to the directed helical line \( \bar{p}^{(j)} \), placed on the basic cylinder \( C^{(j)} \) with radius \( r^{(j)}_0 \). The following condition is accomplished for these points [2, 6]

\[
u^{(j)} \sin \xi^{(j)} = p^{(j)} \vartheta^{(j)}.
\]

(18)

Practical realization of this condition, where the undercutting points appear on the helicoid, is prevented, if the dedendum circle surfaces of the conic involute helical teeth (when the gears are designed) are placed above the basic cylinders \( C^{(j)} \).

Conic Archimedean helicoid. Because of the fact that the condition of distribution for the conic Archimedean helicoid is from the type:

\[
\begin{align*}
0 \neq r^{(j)}_0 \\
p^{(j)} \neq 0
\end{align*}
\]

then the normal vector \( \mathbf{n}^{(j)}_1 \) has projections

\[
\begin{align*}
n^{(j)}_1 &= P^{(j)} \cos \vartheta^{(j)} + U^{(j)} \cos \xi^{(j)} \sin \vartheta^{(j)}, \\
n^{(j)}_2 &= P^{(j)} \sin \vartheta^{(j)} + U^{(j)} \cos \xi^{(j)} \cos \vartheta^{(j)}, \\
n^{(j)}_3 &= U^{(j)} \sin \xi^{(j)}, \quad P^{(j)} = \pm p^{(j)} \sin \xi^{(j)}
\end{align*}
\]

which indicates that this vector is defined for each point \( N^{(j)} \) from \( \Sigma^{(j)}_1 \), i.e. the condition (15) is not accomplished for each point \( N^{(j)} \).

2.3. Synthesis of the surface of action/mesh region of the spatial rack drives

The action surface, considered as a locus of the lines of the tooth contact in the fixed space, defines conjugate tooth surfaces of the second movable link – tooth rack. Based on the realized study and on the developed algorithms for synthesis of three types spatial rack drives – cylindrical with cylindrical linear rotating helicoids, face (with face linear rotating helicoids) and conic (with conic linear rotating helicoids) [1] are written three computer programs with analogical (typified) structure and organization of the calculated process. In the study of the developed programs illustrate the cases of the spatial conic rack drives synthesis. The offered here study is consistent with shown in Fig. 4 symbols.

Conic convolute rack mechanism. It is known, that conic helicoids \( \Sigma^{(j)} \) ( \( j = 1, 2 \) ) can be represented as cylindrical helical surfaces with constant helical parameter \( p^{(j)} = p^{(j)}_1 \pm p^{(j)}_1 \cot \xi^{(j)} = \text{constant} \).

Hence, for these surfaces is valid the equation of meshing [1, 7]:

\[
\tan \Delta = \frac{p^{(j)}_1 - j \_S \cos \vartheta^{(j)}}{j \_S \sin \vartheta^{(j)}} = \text{constant},
\]

(20)

where \( j \_S \) is the acute angle between the vector’s direction of the gear rack and the pinion.

Fig.4. Geometric-kinematic scheme of spatial rack drive having rotating linear conic helicoids

Then the action surface of the conic convolute helicoid is described by using the equations (2) and (16) (written in the fixed coordinate system \( S(O,x,y,z) \)) and the analytical type (20) of equation of meshing (see Fig. 5). The after remodeling we obtain:

\[
\begin{align*}
x^{(j)} &= r^{(j)}_0 \cos \vartheta^{(j)} \pm U^{(j)} \sin \vartheta^{(j)}, \\
y^{(j)} &= r^{(j)}_0 \sin \vartheta^{(j)} \mp U^{(j)} \cos \vartheta^{(j)}, \\
z^{(j)} &= p^{(j)} \vartheta^{(j)} \pm U^{(j)} \cos \xi^{(j)}, \\
H^{(j)} \sin \vartheta^{(j)} + U^{(j)} \cos \xi^{(j)} \cos \vartheta^{(j)} &= \frac{p^{(j)}_1 - j \_S \cos \vartheta^{(j)}}{j \_S \sin \vartheta^{(j)}},
\end{align*}
\]

\[ j \_S \]

(21)

where \( \varphi_j \) – parameter of meshing; \( \vartheta^{(j)} = \vartheta^{(j)} + \varphi_j \), \( U^{(j)} \neq 0 \).
The conjugated with $\Sigma_1^{(j)}$ $(j = 1, 2)$ tooth surfaces $\Sigma_2^{(j)}$ $(j = 1, 2)$ of the tooth rack $i = 2$ are obtained from the system (21), after writing it in the fixed coordinate system $S_2(O_2, x_2, y_2, z_2)$, firmly connected with link $i = 2$:

$$x_2^{(j)} = r_0^{(j)} \cos \Theta^{(j)} \pm U^{(j)} \sin \Theta^{(j)},$$
$$y_2^{(j)} = r_0^{(j)} \sin \Theta^{(j)} \mp U^{(j)} \cos \Theta^{(j)} + j_{2i}\varphi \sin \Sigma_r,$n
$$z_2^{(j)} = p^{(j)} \varphi^{(j)} \pm U^{(j)} \cos \xi^{(j)} + j_{2i}\varphi \cos \Sigma_r,$$

$$\mp h^{(j)} \sin \Theta^{(j)} + U^{(j)} \cot \xi^{(j)} \cos \Theta^{(j)} = = \frac{p^{(j)} - j_{2i}\cos \Sigma_r}{j_{2i} \sin \Sigma_r}.$$  

Conic Archimedean rack mechanism. The analytical representation of the action surface of the conic Archimedean rack (see Fig. 6) drive is obtained from the equation systems (22) after a substitution $r_0^{(j)} = 0$, whence $h^{(j)} = p^{(j)}$, i.e.

$$x^{(j)} = \pm U^{(j)} \sin \Theta^{(j)},$$
$$y^{(j)} = U^{(j)} \cos \Theta^{(j)} + j_{2i}\varphi \sin \Sigma_r,$n
$$z^{(j)} = p^{(j)} \varphi^{(j)} \pm U^{(j)} \cos \xi^{(j)} + j_{2i}\varphi \cos \Sigma_r,$$

$$\pm \frac{p^{(j)} \sin \Theta^{(j)} + U^{(j)} \cot \xi^{(j)} \cos \Theta^{(j)}}{U^{(j)}} = = \frac{p^{(j)} - j_{2i}\cos \Sigma_r}{j_{2i} \sin \Sigma_r}.$$  

Conic involute rack mechanism. From equations systems (49) after substituting $h^{(j)} = 0$, we obtain the analytical form of action surfaces (see Fig. 7) of the studied involute rack mechanism.

$$x^{(j)} = r_0^{(j)} \cos \Theta^{(j)} \pm U^{(j)} \sin \Theta^{(j)},$$
$$y^{(j)} = U^{(j)} \sin \Theta^{(j)} + U^{(j)} \cos \Theta^{(j)},$$
$$z^{(j)} = p^{(j)} \varphi^{(j)} \pm U^{(j)} \cos \xi^{(j)} + j_{2i}\varphi \cos \Sigma_r,$$

$$\cot \xi^{(j)} \cos \Theta^{(j)} = \frac{p^{(j)} - j_{2i}\cos \Sigma_r}{j_{2i} \sin \Sigma_r}.$$  

Bellow (25) is written in the fixed coordinate system, connected with the link $i = 2$. Thus the active tooth surfaces $\Sigma_2^{(j)}$ $(j = 1, 2)$ of the spatial involute conic rack drive are obtained.
x^{(j)}_2 = r_0^{(j)} \cos \varphi^{(j)} + U^{(j)} \sin \varphi^{(j)},
\quad y^{(j)}_2 = r_0^{(j)} \sin \varphi^{(j)} + U^{(j)} \cos \varphi^{(j)} + f_{2j} \varphi \cos \Sigma_r,
\quad z^{(j)}_2 = p^{(j)} \varphi^{(j)} + U^{(j)} \cos \varphi \cos \Sigma_r + f_{2j} \varphi \cos \Sigma_r, \quad (26)
\quad \cot \varphi^{(j)} \cos \varphi^{(j)} = \frac{p^{(j)} - f_{2j} \cos \Sigma_r}{f_{2j} \sin \Sigma_r}.

2.3.1. Analysis of the geometry of the conic linear rack mechanism

Here, the upper indexes "j" will be omitted when this analysis is performed. The analytical type of the action surfaces of the spatial conic convolute rack mechanism is described by the equation systems (21). Using the third equation of the same system, for the curvilinear coordinate $\varphi$ of the conic convolute helicoid $\Sigma$ we can write:

$$\varphi = \frac{z + u \cos \varphi \xi}{p + p \cos \varphi \xi} = f_{1}(z,u). \quad (27)$$

$$\varphi = \frac{z + u \cos \varphi \xi}{p + p \cos \varphi \xi} = f_{1}(z,u). \quad (27)$$

Let multiply both sides of the first equation of (29) with $\sin \varphi$ and the obtained result is reduced to the quadratic equation regarding to $\cos \varphi$, i.e.:

$$(x^2 + y^2) \cos^2 \varphi - 2r_0 \cos \varphi + r_0^2 = 0. \quad (30)$$

The solution of (30) is of the type:

$$\cos \varphi = \frac{r_0 \pm \sqrt{x^2 + y^2 - r_0^2}}{x^2 + y^2} \quad (31)$$

From (31) it follows:

$$(\cos \varphi)_{1,2} = f_2(x,y), \quad (32)$$

Substituting (32) in the second equality of (29) we obtain the following equation:

$$F(x,y) = 0, \quad (33)$$

which is the analytical description of the action surface in the most common case of spatial conic linear rack mechanism – the convolute one. The equality (33) shows that the action surface of these type rack mechanisms is a cylindrical surface with generatrices, parallel to the $z$-coordinate axis $Oz$ of the fixed coordinate system $S(O,x,y,z)$.

The conic Archimedean and involute rack drives are particular cases of the conic convolute rack mechanism. Their action surfaces are described analytically by the system (21), after substituting $r_0 = 0$ (for the Archimedean rack drive) and $h = 0$ (for involute rack mechanism). Hence, in the most common case their action surfaces/mesh regions have characteristics, similar to those characteristics of the conic rack drive. In other words, the conic, Archimedean and involute rack mechanisms have action surfaces that are related (in geometric view point) to the cylindrical (by form) action surfaces of the conic convolute rack drive.

For the case of rack mechanism, which rotating link has conic Archimedean helicoids, the equalities (31) and (32) are:

$$\cos \varphi = \frac{r_0 \pm \sqrt{x^2 + y^2 - r_0^2}}{x^2 + y^2} \quad (31)$$

Then the mesh region is:
The spatial conic rack drive has an action surface, for which the parameter of meshing is $h = 0$, when the rotating link of the conic rack mechanism is equipped with conic involute helicoids. For this case the action surface is described by the equation systems (25). There the equation of meshing is of the form:

$$\cos \theta = \cos(\theta + \varphi_i) = \frac{p - j_2 \cos \Sigma_r \tan \xi}{j_2 \sin \Sigma_r} = \text{constant}.$$  

From (36) follows that when the parameter of meshing $\varphi_i = \text{constant}$ has fixed value, the curvilinear coordinate $\theta$ keeps the constant value for all points from a single contact line, i.e. the contact line of this type conic linear rack mechanisms is a straight line. Thus, if the contact lines of the cylindrical action surface, representing its directrices, are transformed into a straight line, then the action surfaces from cylindrical ones are transformed into a plane.

3. CONCLUSION

The present study gives a brief survey of the approach to used design mesh region mathematical model of spatial rack transmission synthesis. This approach has been applied to model different types of spatial rack mechanisms. The application of the model in convolute, Archimedean and involute rack mechanisms synthesis is also illustrated. Algorithms and computer graphics for synthesis and design of the spatial rack mechanisms are elaborated.

The discussed above mechanisms are suitable for implementation as actuators in various fields of techniques. Of particular interest is their incorporation into the constructions of bio-robots [8], as an alternative of spatial hyperboloid gears [9].

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